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CARTAN–FUBINI TYPE EXTENSION OF HOLOMORPHIC MAPS FOR FANO MANIFOLDS OF PICARD NUMBER 1

Jun-Muk HWANG [a,1], **Ngaiming MOK** [b,2]

[a] *Korea Institute for Advanced Study, 207-43 Cheongryangri-dong, Seoul 130-012, South Korea*
[b] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*

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In the study of manifolds having the geometric structure modeled on Hermitian symmetric spaces [2] and the deformation rigidity of irreducible Hermitian symmetric spaces of the compact type [3], the following result of Ochiai [12] played an essential role.

THEOREM (Ochiai). – *Let X be an irreducible Hermitian symmetric space of the compact type of rank $\geqslant 2$; X has a natural G-structure where G is the reductive Levi factor of the isotropy subgroup of a base point of X. Let $U_1, U_2 \subset X$ be two connected open sets and $\varphi : U_1 \to U_2$ be a biholomorphism preserving the G-structure. Then φ can be extended to a biholomorphic automorphism of X.*

This result was generalized to other rational homogeneous spaces by Yamaguchi [13], where the statement holds with 'G-structure' replaced by a natural geometric structure on the homogeneous space. Their proof relies on the vanishing of certain Lie algebra cohomology groups. Since this result is very useful in the study of many geometric problems on rational homogeneous spaces, one may ask whether a more geometric proof can be given using only rational curves, so that it can be generalized to some non-homogeneous projective manifolds. This was partially achieved in Secions 3 and 4 of [5], where the authors were able to give a proof of the above result of Ochiai and Yamaguchi, via the deformation theory of rational curves and basic theory of differential systems, without using Lie algebra cohomology. Still, it was unsatisfactory in the sense that one has to use group actions to analytically continue φ to the whole X, so the proof works only for the homogeneous manifolds.

In this paper, we overcome this by introducing analytic continuations along special families of rational curves and give a proof which can work for a large class of Fano manifolds of Picard number 1.

To state our result, it is necessary to define a natural 'geometric structure' on a Fano manifold of Picard number 1. This is given by tangent vectors to standard rational curves. Roughly speaking, a standard rational curve is an immersed $\mathbf{P}_1$ in the Fano manifold X whose normal bundle contains only $\mathcal{O}(1)$ and $\mathcal{O}$ factors. Such curves exist by a result of Mori [11]. Choosing a maximal irreducible family $\mathcal{H}$ of standard rational curves, we define the variety of $\mathcal{H}$-tangent $\mathcal{C} \subset \mathbf{P}T(X)$ as the collection of tangent vectors to standard rational curves belonging to $\mathcal{H}$ (see Section 1 for details). This corresponds to our geometric structure on X. In the case of a

E-mail addresses: jmhwang@ns.kias.re.kr (J.-M. Hwang), nmok@hkucc.hku.hk (N. Mok).

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rational homogeneous space X of Picard number 1, the lines on X under the minimal projective embedding of X are standard rational curves and the associated $\mathcal{C}$ corresponds to the natural geometric structure on X. In other words, the condition on φ "preserving the G-structure" in the above theorem of Ochiai can be replaced by "whose differential sends $\mathcal{C}|_U$ to $\mathcal{C}|_{U'}$". Our main theorem is a generalization of Ochiai's theorem in this sense. We can give a rough outline of the statement of the main theorem as follows. See Theorem 1.2 for the precise statement.

MAIN THEOREM. – *Let X be a Fano manifold of Picard number* 1. *Suppose there exists a family of standard rational curves $\mathcal{H}$ such that the associated $\mathcal{C} \subset \mathbf{P}T(X)$ satisfies certain conditions which hold for many examples as given in Section* 1. *Let X' be any Fano manifold of Picard number* 1 *and $\mathcal{H}'$ be a family of standard rational curves on X'. Given any connected open subsets $U \subset X, U' \subset X'$ with a biholomorphic map $\varphi : U \to U'$ such that the differential $\varphi_* : \mathbf{P}T_x(X) \to \mathbf{P}T_{\varphi(x)}(X')$ sends each irreducible component of $\mathcal{C}|_U$ to an irreducible component of $\mathcal{C}|_{U'}$ biholomrophically, there exists a biholomorphic map $\Phi : X \to X'$ such that φ is the restriction of Φ to U.*

This result is stronger than Ochiai's even for the irreducible Hermitian symmetric space X in the sense that we need not assume that X' is *a priori* biholomorphic to X.

When both X and X' are hypersurfaces of low degree in the projective space, Main Theorem can be proved using the result of Jensen and Musso [8] which completed a study initiated by E. Cartan and G. Fubini. Although the method of proof and basic ideas are completely different, the origin of this type of problem goes back to E. Cartan and G. Fubini, and we name the extension of the above kind 'Cartan–Fubini type extension'.

We expect that there are many applications of the Cartan–Fubini type extension property. As a matter of fact, our works [2–4] can be viewed as applications of known special cases of Cartan–Fubini type extension for rational homogeneous spaces. In this article the extension property will be applied to large classes of Fano manifolds of Picard number 1 to prove local rigidity of generically finite morphisms, which we will explain at the end of Section 1, after giving precise statement of the main theorem, Theorem 1.2, and some examples.

The proof of Theorem 1.2 will be given in Sections 2–4. Section 2 and Section 3 are the main part of the analytic continuation. Our analytic continuation is different from the classical one in the sense that it should by carried out only along the rational curves involved. For this, we introduce the concept of 'parametrized analytic continuation'. The proof will be finished in Section 4 by proving first that the map can be extended to a bimeromorphic map and then that it cannot have a ramification locus.

A few words on the terminology are in order. When we say an open set, it is in the classical topology, not Zariski topology, unless it is specifically said so. By a generic point of an analytic variety, we mean a point outside the union of countably many proper analytic subvarieties. A variety is not necessarily irreducible, but has only finitely many components.

1. Statement and examples of the main result

We start with defining some terms that we are going to use throughout the article. We will skip most of the proofs of standard facts, refering 1.1 of [5] and II.2 of [10] for further details.

A rational curve $h : \mathbf{P}_1 \to X$ on a complex manifold X is called a *standard rational curve*, if $h^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some nonnegative integers p, q. In this case, h is necessarily a holomorphic immersion and birational. From $H^1(\mathbf{P}_1, h^*T(X)) = 0$, the space $Hom(\mathbf{P}_1, X)$ of morphisms from $\mathbf{P}_1$ to X is smooth at the point $[h]$ and the tangent space is $H^0(\mathbf{P}_1, h^*T(X))$. Let $\mathcal{H}$ be an irreducible component of $Hom(\mathbf{P}_1, X)$ containing a standard rational curve. Then a generic point of $\mathcal{H}$ is a standard rational curve. An irreducible component $\mathcal{H}$ of $Hom(\mathbf{P}_1, X)$ will be called a *standard component* if a generic member of $\mathcal{H}$ is a standard rational curve. The following properties of standard rational curves will be useful:

LEMMA 1.1. – *Let $h : \mathbf{P}_1 \to X$ be a standard rational curve. Then*:

(1) *The image of deformations of h cover an open neighborhood of $h(\mathbf{P}_1)$ in X.*

(2) *Let h_t be a deformation of $h = h_0$ parametrized by the disc $\Delta := \{t \in \mathbf{C}, |t| < 1\}$. Suppose the deformation h_t fixes two points, namely, for two distinct points $o, \infty \in \mathbf{P}_1$ and for all t, $h_t(o) = h_0(o)$ and $h_t(\infty) = h_0(\infty)$. Then h_t is a trivial deformation in the sense that $h_t(s) = h_0(s)$ for all $s \in \mathbf{P}_1$.*

(3) *For any given subvariety of codimension 2 in X, some deformation of h has image disjoint from it.*

Proof. – (1) follows from $H^1(\mathbf{P}_1, h^*T(X)) = 0$ and the fact $h^*T(X)$ is generated by global sections; (2) follows from the fact that the normal sheaf $h^*T(X)/T(\mathbf{P}_1) \cong [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ cannot have sections vanishing at two distinct points. For the proof of (3), see the proof of Proposition 12 of [4]. □

Given a standard component $\mathcal{H}$, the natural action of the automorphism group of $\mathbf{P}_1$ gives $\mathcal{H}$ a structure of $\mathbf{PGL}_2$-principal bundle over an analytic space $\mathcal{K}$. The graphs of the elements of $Hom(\mathbf{P}_1, X)$ induces a $\mathbf{P}_1$-bundle $\mathcal{U}$ over $\mathcal{K}$, with natural universal family morphisms $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$. Let $\mathcal{K}^o \subset \mathcal{K}$ be the Zariski-open subset consisting of standard rational curves and $\mathcal{U}^o := \rho^{-1}(\mathcal{K}^o)$ be the universal family over $\mathcal{K}^o$. Then $\mathcal{K}^o$ is a complex manifold of dimension $n + p - 1$. By associating the tangent vectors to standard rational curves, we define the tangent morphism $\tau : \mathcal{U}^o \to \mathbf{P}T(X)$, which is a holomorphic immersion. Let $\mathcal{C} \subset \mathbf{P}T(X)$ be the closure of the image of τ. $\mathcal{C}$ will be called the *variety of $\mathcal{H}$-tangents*, or *variety of rational tangents* if the choice of $\mathcal{H}$ is clear. For a point $x \in X$, we call $\mathcal{C}_x := \mathcal{C} \cap \mathbf{P}T_x(X)$ *the variety of $\mathcal{H}$-tangents at x*. We define $\mathcal{U}_x := \mu^{-1}(x)$ and $\mathcal{U}^o_x = \mathcal{U}_x \cap \mathcal{U}^o$. Let $\tau_x : \mathcal{U}^o_x \to \mathbf{P}T_x(X)$ be the restriction of τ. For a generic point $x \in X$, $\mathcal{C}_x$ is equal to the closure of the image of τ_x.

The foliation on $\mathcal{U}^o$ defined by the fibers of ρ induces a multi-valued foliation $\mathcal{F}$ on a Zariski-open set of $\mathcal{C}$ by the immersion $\tau : \mathcal{U}^o \to \mathcal{C}$. $\mathcal{F}$ will be called the *tautological foliation* on $\mathcal{C}$. This name is not precise in the sense that $\mathcal{F}$ may be multi-valued. However, in the case we deal with in this article, it will be a univalent foliation.

When X is a projective manifold, $\mathcal{K}$ is a quasi-projective scheme which is the semi-normalization of the subvariety of the Chow variety corresponding to the images of elements of $\mathcal{H}$ and $\rho : \mathcal{U} \to \mathcal{K}$ is induced by the universal family over the Chow variety. See II.2 of [10] for details. It follows that we can naturally compactify $\mathcal{K}$ and $\mathcal{U}$ to projective varieties and the universal family morphisms ρ and μ can be extended. For projective X, we will use the same symbols $\mathcal{K}, \mathcal{U}$ to denote these projective varieties. $\rho : \mathcal{U} \to \mathcal{K}$ is no longer a $\mathbf{P}_1$-bundle, but just its generic fiber is $\mathbf{P}_1$. Mostly, we will work with $\mathcal{K}$ instead of $\mathcal{H}$, because we only use the property of the image of $h : \mathbf{P}_1 \to X$. For simplicity, we will call the image curve $C = h(\mathbf{P}_1)$ simply as a standard rational curve.

Now let X and X' be Fano manifolds of Picard number 1. By Mori's bend-and-break trick [11], X and X' contain standard rational curves. Let $\mathcal{H}$ (resp. $\mathcal{H}'$) be a standard component and $\mathcal{C}$ (resp. $\mathcal{C}'$) be the variety of $\mathcal{H}$-tangents (resp. $\mathcal{H}'$-tangents) which has fiber dimension p (resp. p'). We say that *Cartan–Fubini type extension* holds for the pair $(X, \mathcal{H})$, if for any choice of $X', \mathcal{H}'$ with $p = p'$ and any connected open subsets $U \subset X, U' \subset X'$ with a biholomorphic map $\varphi : U \to U'$ such that the differential $\varphi_* : \mathbf{P}T_x(X) \to \mathbf{P}T_{\varphi(x)}(X')$ sends each irreducible component of $\mathcal{C}_x$ to an irreducible component of $\mathcal{C}'_{\varphi(x)}$ for all generic $x \in U$, there exists a biholomorphic map $\Phi : X \to X'$ such that φ is the restriction of Φ to U. Note that the condition on φ does not require $\mathcal{C}'_{\varphi(x)} = \varphi_*(\mathcal{C}_x)$. In other words, *a priori* it may happen that $\mathcal{C}'_{\varphi(x)}$ contains other components different from those of $\varphi_*(\mathcal{C}_x)$. This point will be essential in the application below, Theorem 1.4. Our main result is the following:

THEOREM 1.2. – *Let X be a Fano manifold with Picard number* 1*. Suppose there exists a standard component $\mathcal{H}$ with $p, q > 0$ such that for a generic point $x \in X$, the Gauss map for each irreducible component of $\mathcal{C}_x$ at x as a projective subvariety of $\mathbf{P}T_x(X)$ is generically finite. Then Cartan–Fubini type extension holds for $(X, \mathcal{H})$.*

There are many examples of Fano manifolds where the conditions for Theorem 1.2 hold. The condition on the Gauss map holds, if it holds for some component of $\mathcal{C}_x$ at generic $x \in X$ by the irreducibility of $\mathcal{C}$. By Zak's result [14] or its weaker version [1], this condition is satisfied if $\mathcal{C}_x$ is smooth and not linear. Suppose $\mathcal{H}$-curves are lines under a projective embedding of X. Then the smoothness of $\mathcal{C}_x$ at generic $x \in X$ is well-known and the condition $p, q > 0$ is equivalent to $3 \leqslant c_1(X) \leqslant \dim(X)$. So Theorem 1.2 works in the following two cases:

(1) Rational homogeneous space G/P of Picard number 1 different from the projective space. $\mathcal{K}$ is the set of lines under the minimal projective embedding. $\mathcal{C}_x$ is smooth and not linear.

(2) Smooth linearly nondegenerate complete intersections $X \subset \mathbf{P}_N$ of dimension $\geqslant 2$ and of multi-degree $(d_1, \ldots, d_l)$ with $1 < d_1 + \cdots + d_l \leqslant N - 2$. $\mathcal{K}$ is the set of lines of $\mathbf{P}_N$ lying on X. $\mathcal{C}_x$ is a smooth complete intersection for generic x. Defining equations of $\mathcal{C}_x$ can be obtained by differentiating the defining equations of X.

The following is an example where the standard rational curves are not lines under a projective embedding:

(3) Let X be the moduli space of stable bundles of rank 2 with a fixed determinant of odd degree over a smooth projective curve of genus $\geqslant 5$. Through a generic point of X, there exists a standard rational curve arising from Hecke correspondence, called a Hecke curve. For the corresponding standard component, $\mathcal{C}_x$ is a ruled surface which is nondegenerate and smooth in $\mathbf{P}T_x(X)$ for generic $x \in X$. See [7] for details.

In the statement of Theorem 1.2, the condition that $q > 0$ is necessary. In fact, if $q = 0$, which is the case for the projective space, the condition on φ of preserving varieties of rational tangents is void and φ can be just any local biholomorphic map.

On the other hand, the condition $p > 0$ is restrictive. There are many Fano manifolds with Picard number 1 such that $p = 0$ for all standard components with $q > 0$. Most notably, smooth hypersurfaces of degree n in $\mathbf{P}_{n+1}$ belong to this case as well as all Fano threefolds of Picard number 1 with index 2. But we do not know whether there exists an example with $p = 0$ for which the Cartan–Fubini type extension property does not hold.

Our proof heavily depends on the condition $p > 0$. The condition on the Gauss map will be used only for the following result proved in 3.1 of [5].

PROPOSITION 1.3. – *Assume that $(X, \mathcal{H})$ satisfies the assumptions of Theorem* 1.2*. Then the tangent morphism $\tau : \mathcal{U}^o \to \mathcal{C}$ is birational. Furthermore, for any choice of Fano manifold X' with Picard number* 1*, a standard component $\mathcal{H}'$ with $\mathcal{C}' \subset \mathbf{P}T(X')$ having fiber dimension p over X', and any connected open subsets $U \subset X, U' \subset X'$, if there exists a biholomorphic map $\varphi : U \to U'$ satisfying $\varphi_*(\mathcal{C}_x) \subset \mathcal{C}'_{\varphi(x)}$ for all generic $x \in U$, then for any member C of $\mathcal{K}$, $\varphi(C \cap U)$ is contained in $C' \cap U'$ for some member C' of $\mathcal{K}'$. In other words, φ sends local pieces of $\mathcal{H}$-curves to local pieces of $\mathcal{H}'$-curves.*

Proof. – The birationality of τ is stated in Corollary 3.1.5 of [5], where it is proved that the tautological foliation is uniquely determined by the variety of minimal rational tangents if the Gauss map condition is satisfied. The second statement is an immediate consequence of this. □

The proof of Theorem 1.2 will be given in Sections 2–4. We want to finish this section with an application. The Cartan–Fubini type extension property implies the rigidity of generically finite morphisms in the following sense:

THEOREM 1.4. – *Let $(X_0, \mathcal{H}_0)$ be a Fano manifold of Picard number* 1 *with the Cartan–Fubini type extension property. Let Y be any complete variety and $\pi : \mathcal{X} \to \Delta := \{t \in \mathbf{C}, |t| < 1\}$ be a regular family of Fano manifolds of Picard number* 1 *such that $X_0 = \pi^{-1}(0)$. Then for any surjective morphism $f : Y \times \Delta \to \mathcal{X}$ over Δ such that the restriction $f_t : Y \to X_t = \pi^{-1}(t)$ is generically finite for each $t \in \Delta$, there exists $\varepsilon > 0$ and a unique holomorphic family of biholomorphic morphisms $g_t : X_0 \to X_t$ for $|t| < \varepsilon$ satisfying $f_t = g_t \circ f_0$.*

COROLLARY 1.5. – *Given any complete varieties X and Y of the same dimension, let Hol(Y, X) be the set of surjective holomorphic maps from Y to X. Then for any fixed Y and any Fano manifold X of Picard number* 1 *having the Cartan–Fubini type extension property with respect to some choice of a standard component, Hol(Y, X) is countable up to automorphisms of X. Furthermore there exist only countably many such Fano manifolds X, for which Hol$(Y, X) \neq \emptyset$.*

For the proof of Theorem 1.4, we need to recall some results from [4]. Let Y be a projective manifold and $y \in Y$ be a point. In Section 1 of [4], we define the notion of a variety of distinguished tangents. Roughly speaking, an irreducible subvariety of $\mathbf{P}T_y(Y)$ is a variety of distinguished tangents if it is the closure of tangent vectors to a family of curves passing through y which corresponds to a stratum of a natural stratification of the Hilbert scheme of curves through y. We refer to [4] for precise definitions. What we need here is the fact that there are only countably many varieties of distinguished tangents in $\mathbf{P}T_y(Y)$, which is an immediate consequence of the definition. We also need the following proposition:

PROPOSITION 1.6 (Proposition 3 in [4]). – *Let $f : Y \to X$ be a generically finite surjective morphism from a projective manifold Y to a Fano manifold X of Picard number* 1. *Choose a standard component on X and let $\mathcal{C} \subset \mathbf{P}T(X)$ be the variety of rational tangents. Then for any generic point $y \in Y$, each irreducible component of the subvariety $df_y^{-1}(\mathcal{C}_{f(y)}) \subset \mathbf{P}T_y(Y)$ is a variety of distinguished tangents.*

In [4], this was stated for a finite morphism f and varieties of "minimal rational tangents" on X. But the proof works equally well for the general case stated above.

Proof of Theorem 1.4. – A standard rational curve $h : \mathbf{P}_1 \to X_0$ can be viewed as a standard rational curve of $\mathcal{X}$. By Lemma 1.1(1), there exists a family $h_t : \mathbf{P}_1 \to X_t$ for $|t| < \varepsilon$ for some $\varepsilon > 0$, which is a standard rational curve in each X_t. Let $\mathcal{H}_t$ be the standard component of $Hom(\mathbf{P}_1, X_t)$ containing h_t. Let $\mathcal{X}_\varepsilon = \pi^{-1}(\{|t| < \varepsilon\})$.

Let $\mathcal{H}$ be the standard component of $Hom(\mathbf{P}_1, \mathcal{X}_\varepsilon)$ containing h_t's and let $\mathcal{C} \to \mathbf{P}T(\mathcal{X}_\varepsilon)$ be the variety of $\mathcal{H}$-tangents. Since all images of elements of $Hom(\mathbf{P}_1, \mathcal{X})$ are contained in the fibers of π, $\mathcal{C}$ is contained in the subbundle $\mathbf{P}T^\pi$ of $\mathbf{P}T(\mathcal{X})$ where T^π denotes the relative tangent bundle of π. Let $\mathcal{C}_t \subset \mathbf{P}T(X_t)$ be the variety of $\mathcal{H}_t$-tangents for $|t| < \varepsilon$. We claim that the limit of $\mathcal{C}_t$ as t approaches 0, contains $\mathcal{C}_0$ as a component, although the limit may have components different from $\mathcal{C}_0$. Since $\mathcal{C}$ is locally an immersed submanifold near the point corresponding to h and $\mathcal{C}$ is irreducible, there exists some $\varepsilon' < \varepsilon$ so that $\mathcal{C} \cap \mathbf{P}T(X_t)$ is irreducible for $0 < |t| < \varepsilon'$. In particular, $\mathcal{C} \cap \mathbf{P}T(X_t)$ is exactly $\mathcal{C}_t$, for $0 < |t| < \varepsilon'$ and $\mathcal{C}_0$ is contained in the closure of the union of $\mathcal{C}_t$ for $0 < |t| < \varepsilon'$, as claimed.

Choose a small open set $U^* \subset Y$ and shrink ε' if necessary, so that $f_t|_{U^*}$ is biholomorphic for small t and the image $f_t(U^*)$ is contained in the open set covered by the union of images of $\mathcal{H}_t$, $|t| < \varepsilon'$. For each $y \in U^*$, let $\mathcal{C}_{f_t(y)}$ be the variety of rational tangents at $f_t(y)$ associated to

$\mathcal{H}_t, 0 < |t| < \varepsilon'$. Then the closure of the union of $\{\mathcal{C}_{f_t(y)}, 0 < |t| < \varepsilon'\}$ contains $\mathcal{C}_{f_0(y)}$, the variety of rational tangents at $f_0(y)$ for $\mathcal{H}_0$. By Proposition 1.6, $\{df_t^{-1}(\mathcal{C}_{f_t(y)}), 0 < |t| < \varepsilon'\}$ gives a family of varieties of distinguished tangents in $\mathbf{P}T_y(Y)$. Since there are only countably many varieties of distinguished tangents in $\mathbf{P}T_y(Y)$, $f_t^{-1}(\mathcal{C}_{f_t(y)})$ is independent of t and $f_0^{-1}(\mathcal{C}_{f_0(y)})$ is the union of some components of $f_t^{-1}(\mathcal{C}_{f_t(y)}), t \neq 0$. The biholomorphic map $\varphi_t := f_t \circ f_0^{-1}$ from $f_0(U^*)$ to $f_t(U^*)$ sends each component of $\mathcal{C}_{f_0(y)}$ to a component of $\mathcal{C}_{f_t(y)}$. By the Cartan–Fubini type extension property, it can be extended to a biholomorphic map $g_t : X_0 \to X_t$ satisfying $f_t = g_t \circ f_0$. Since π is a family of Kähler manifolds and φ_t depends holomorphically on t, the graphs of $g_t, 0 < |t| < \varepsilon'$, converge to the graph of g_0 and $\{g_t, |t| < \varepsilon'\}$ form a holomorphic family. □

Remark. – Although we do not know whether Cartan–Fubini type extension holds for the case of $p = 0$, an analogue of Theorem 1.4 for the case of $p = 0$ is proved in [6], by using a completely different method which cannot work for the case $p > 0$.

2. Analytic continuation along standard rational curves

For the biholomorphic map $\varphi : U \to U'$ in the statement of Theorem 1.2, we will say that φ preserves varieties of rational tangents. For the proof of Theorem 1.2 we will have to deal with locally defined meromorphic maps which preserve varieties of rational tangents at generic points. More precisely, let $\Omega \subset X$ be a connected open set and $\varphi : \Omega \to X'$ be a meromorphic map. We say that φ *preserves varieties of rational tangents* if and only if (a) φ is of maximal rank at a generic point $x \in \Omega$ and (b) for such $x \in \Omega$ we have $\varphi_* \mathcal{C}_x \subset \mathcal{C}'_{\varphi(x)}$, i.e., φ_* sends each component to $\mathcal{C}_x$ to a component of $\mathcal{C}'_{\varphi(x)}$.

Theorem 1.2 will be proved by constructing an analytic continuation of φ. This analytic continuation is different from the classical one, in the sense that we have to carry it out only along standard rational curves. Let $C \subset X$ be a $\mathcal{K}^o$-curve intersecting U. We want to get an analytic continuation of φ along paths lying on C. This analytic continuation needs not be univalent because C is not necessarily smooth. Moreover we want to repeat this process along other standard rational curves intersecting U. For this reason, it is convenient to introduce the notion of parametrized analytic continuation along a holomorphic map from a complex space into X.

Let x_0 be a point on X and φ be a germ of meromorphic map into X' at x_0 preserving varieties of rational tangents. Let S be a complex space and $s_0 \in S$ be a base point. Let $\lambda : S \to X$ be a holomorphic map such that $\lambda(s_0) = x_0$. By the *parametrized analytic continuation of* φ *along* λ we mean a germ of meromorphic map F along $\Sigma := \mathrm{Graph}(\lambda) \subset S \times X$ such that:

(a) denoting by $pr_X : S \times X \to X$ the canonical projection onto the second factor, the germ of F at $(s, \lambda(s))$ agrees with $pr_X^* \nu$ for some germ of meromorphic map ν into X' at $\lambda(s) \in X$ for each $s \in S$;

(b) the germ of F at (s_0, x_0) agrees with $pr_X^* \varphi$.

We will write $\lambda : (S; s_0) \to (X; x_0)$ to indicate that $s_0 \in S$ is the base point, $\lambda(s_0) = x_0$. We sometimes write $(\varphi; x_0)$ for the germ of φ at x_0, and $(F; \Sigma)$ for the germ of F along Σ, etc.

We have analytic continuation of the meromorphic map preserving rational tangents along standard rational curves in the following way:

PROPOSITION 2.1. – *Under the assumptions of Theorem* 1.2, *let x_0 be a point in U and C_0 be a standard rational curve through x_0. Choose a point $u_0 \in \rho^{-1}([C_0])$ satisfying $\mu(u_0) = x_0$. (A choice of u is equivalent to the choice of a local irreducible component of C_0 at x.) Then, there exists an open neighborhood $\mathcal{B}_0$ of* $[C_0]$ *in $\mathcal{K}^o$, so that for $\lambda := \mu|_{\rho^{-1}(\mathcal{B}_0)} : \rho^{-1}(\mathcal{B}_0) \to X$,* *there exists a parametrized analytic continuation of the germ of meromorphic map* $(\varphi; x_0)$ *along* $\lambda : (\rho^{-1}(\mathcal{B}_0); u_0) \to (X; x_0)$.

We will prove three lemmata first.

LEMMA 2.2. – *Let* $\Omega \subset X$ *be a connected open set and* $\varphi : \Omega \to X'$ *be a meromorphic map preserving varieties of rational tangents. Let* $x \in \Omega$ *be a point and* $[C] \in \mathcal{K}^o$ *be a standard rational curve passing through* x. *Choose* $u \in \rho^{-1}([C]) \subset \mathcal{U}^o$ *such that* $\mu(u) = x$. *Then, there exist an open neighborhood* $\mathcal{W}$ *of* u *in* $\mathcal{U}^o$, *an open neighborhood* $\mathcal{B}$ *of* $[C]$ *in* $\mathcal{K}^o$, *together with meromorphic maps* $\varphi^\flat : \mathcal{W} \to \mathcal{U}'$, $\varphi^\# : \mathcal{B} \to \mathcal{K}'$, *such that* $\tau' \circ \varphi^\flat \equiv [d\varphi] \circ \tau$ *and* $\rho' \circ \varphi^\flat \equiv \varphi^\# \circ \rho$. *Moreover, the germs of* $\varphi^\flat$ *at* u *and of* $\varphi^\#$ *at* $[C]$ *are uniquely determined by* φ *and they are of maximal rank at generic points.*

Here and henceforth an open neighborhood is always understood to be connected. As is evident $\tau' : \mathcal{U}' \to \mathbf{P}T(X')$ denotes the analogue of $\tau : \mathcal{U} \to \mathbf{P}T(X)$, etc.

Proof. – Consider

$$\mathcal{U}^o|_\Omega \xrightarrow{\tau} \mathcal{C}|_\Omega \xrightarrow{\varphi_*} \mathcal{C}' \xleftarrow{\tau'} \mathcal{U}'^o.$$

By Proposition 1.3, $\tau : \mathcal{U}^o \to \mathcal{C}$ and $\tau' : \mathcal{U}'^o \to \mathcal{C}'$ are birational immersions. We define $\varphi^\flat$ to be the composition $\tau'^{-1} \circ \varphi_* \circ \tau$, which is a meromorphic map from $\mathcal{U}|_\Omega$ into $\mathcal{U}'$. Let $\mathcal{W}$ be the connected component of $\mathcal{U}|_\Omega$ containing u. By Proposition 1.3, $\varphi^\flat$ sends the fibers of ρ on $\mathcal{W}$ to fibers of ρ', inducing a meromorphic map $\varphi^\# : \mathcal{B} \to \mathcal{K}'$ for some open set $\mathcal{B} \subset \mathcal{K}^o$ containing $[C]$. □

LEMMA 2.3. – *Suppose we are given a connected open set* $\mathcal{B} \subset \mathcal{K}^o$ *and a meromorphic map* $\xi : \mathcal{B} \to \mathcal{K}'$. *For any* $[C] \in \mathcal{K}^o$, *any* $x \in C$ *and* $u \in \mu^{-1}(x) \cap \rho^{-1}([C])$, *there exists at most one germ of meromorphic map* φ *at* x *to* X' *preserving varieties of rational tangents, so that the germ of the induced map* $\varphi^\#$ *at* $[C]$ *with respect to* u *defined in Lemma* 2.2 *agrees with* ξ.

Proof. – Suppose not. We may assume that:

(i) there exist two distinct meromorphic maps $\varphi_1, \varphi_2 : \Omega \to X'$ on some neighborhood Ω of x, both of them preserving rational tangents;

(ii) the induced maps $\varphi_1^\flat$ and $\varphi_2^\flat$ are defined on the same neighborhood $\mathcal{W}$ of u;

(iii) the induced maps $\varphi_1^\#$ and $\varphi_2^\#$ are defined and equal on $\mathcal{B}$.

Let $y \in \Omega$ be a generic point; $\rho(\mu^{-1}(y))$ is a p-dimensional family of standard rational curves through y. Recall that $\varphi_1^\#$ and $\varphi_2^\#$ have maximal rank at generic points. By $\varphi_1^\#$, it will be sent to a p-dimensional family of standard rational curves on X' passing through $\varphi_1(y)$. By $\varphi_2^\#$, it will be sent to a p-dimensional family of standard rational curves passing through $\varphi_2(y)$. But $\varphi_1^\# = \varphi_2^\#$, so we get a p-dimensional family of standard rational curves on X' passing through two distinct points $\varphi_1(y) \neq \varphi_2(y)$. A contradiction to Lemma 1.1(2). □

LEMMA 2.4. – *Suppose we are given a* $\mathcal{K}^o$*-curve* $C \subset X$, *a point* $x \in C$, $u \in \rho^{-1}([C]) \cap \mu^{-1}(x)$, *and a meromorphic map* $\varphi : \Omega \to X'$ *on a neighborhood of* x *preserving rational tangents. Choose* $\mathcal{W}, \mathcal{B}, \varphi^\flat, \varphi^\#$ *as in Lemma* 2.2. *Let* Δ^p *denote the* p*-dimensional polydisc. Given* $y \in C$ *and* $w \in \rho^{-1}([C]) \cap \mu^{-1}(y)$ *with neighborhoods* $y \in D_y$ *in* X *and* $w \in \mathcal{D}_w$ *in* $\mathcal{U}$ *satisfying*:

(i) $\mathcal{D}_w \subset \rho^{-1}(\mathcal{B})$;

(ii) $\mu(\mathcal{D}_w) = D_y$ *and* $\mathcal{D}_w$ *is biholomorphic to* $D_y \times \Delta^p$ *in such a way that the fiber of* $\mu|_{\mathcal{D}_w}$ *over* $z \in D_y$ *corresponds to* $\{z\} \times \Delta^p$;

(iii) $\mathcal{D}_w \cap \mathcal{W} \neq \emptyset$ *and* $D_y \cap \Omega \neq \emptyset$;

there exists a meromorphic map $\varphi_1 : D_y \to X'$ preserving rational tangents, so that $\varphi_1 = \varphi$ on $D_y \cap \Omega$ and the induced maps $\varphi_1^{\#}$ agrees with $\varphi^{\#}$ as germs of meromorphic maps at $[C] \in \mathcal{B}$.

Proof. – Define $\zeta : \rho^{-1}(\mathcal{B}) \to \mathcal{K}'$ by $\zeta := \varphi^{\#} \circ \rho$. Identify $\mathcal{D}_w$ with $D_y \times \Delta^p$. Then choosing a point $v \in \Delta^p$ corresponds to assigning a $\mathcal{K}^o$-curve $C_{x,v}$ to each point x of D_y. Choose a generic $v \in \Delta^p$ so that ζ is holomorphic at a generic point of $D_y \times \{v\}$. This gives a $\mathcal{K}'^o$-curve $C'_{x,v}$ for each $x \in D_y$, defined by the meromorphic map $\zeta_v : D_y \to \mathcal{K}'$ by $\zeta_v(z) = \zeta(z, v)$ for $z \in D_y$. We want to show that the family of curves $C'_{x,v}$ defined by generic choices of $v \in \Delta^p$ has a unique common point and define $\phi_1(x)$ as this common point. To make it precise, we will work with their graphs.

Let $\Theta_v \subset D_y \times \mathcal{U}'$ be defined by:

$$\begin{aligned}\Theta_v &:= (\mathrm{id}, \rho')^{-1}\big(\mathrm{Graph}(\zeta_v)\big)\\ &= \big\{(x, u') \in D_y \times \mathcal{U}', \rho'(u') \in C'_{x,v}\big\}.\end{aligned}$$

Let (id, μ') be the map $D_y \times \mathcal{U}' \to D_y \times X'$ and define:

$$\begin{aligned}\Pi_v &:= (\mathrm{id}, \mu')(\Theta_v)\\ &= \big\{(x, x') \in D_y \times X', x' \in C'_{x,v}\big\}.\end{aligned}$$

Then Π_v is an analytic subvariety of $D_y \times X'$ which is proper over D_y. Consider now the intersection:

$$\begin{aligned}\Pi &:= \bigcap\big\{\Pi_v : v \in \Delta^p, \zeta_v \text{ is holomorphic at a generic point of } D_y\big\}\\ &= \Big\{(x, x') \in D_y \times X', x' \in \bigcap C'_{x,v} \text{ for generic } v \in \Delta^p\Big\}.\end{aligned}$$

Then Π is also proper over D_y. With respect to the canonical projection $D_y \times X' \to D_y$ the fiber of $\Pi \subset D_y \times X'$ over a generic point consists of the intersection of a p-dimensional family of standard rational curves on X'.

Over a generic point $z \in D_y \cap \Omega$, this is exactly the p-dimensional family of standard rational curves passing through $\varphi(z)$, and $\Pi|_{D_y \cap \Omega}$ can be regarded as the graph of $\varphi|_{D_y \cap \Omega}$. So $\Pi|_{D_y \cap \Omega}$ is bimeromorphic over $D_y \cap \Omega$. From the properness of Π over D_y, there exists a unique component of Π which is bimeromorphic over D_y, defining a meromorphic map $\varphi_1 : D_y \to X'$. It certainly satisfies the required properties. □

Proof of Proposition 2.1. – From φ at x_0 and u_0, we get $\mathcal{B}, \varphi^{\flat}, \varphi^{\#}$ as in Lemma 2.2. Since μ is submersive along $\rho^{-1}([C_0])$ by Lemma 1.1(1), we can choose finitely many points $y_i \in C_0$, $w_i \in \rho^{-1}([C_0]) \cap \mu^{-1}(y_i)$ and cover $\rho^{-1}([C_0])$ by finite number of open sets $\mathcal{D}_{w_i}$'s in $\rho^{-1}(\mathcal{B})$ so that $\mathcal{D}_{w_i} \cong D_{y_i} \times \Delta^p$ for suitable D_{y_i}'s covering C_0. Choose $\mathcal{B}_0 \subset \mathcal{B}$ so that $\rho^{-1}(\mathcal{B}_0) \subset \bigcup \mathcal{D}_{w_i}$. By repeatedly applying Lemma 2.4, we obtain analytic continuation $\tilde{\varphi}_i$ of φ on D_{y_i}. This may not be univalent on the open set $\bigcup D_{y_i}$ of X. But its pull-back to $\bigcup \mathcal{D}_{w_i}$ must be univalent by Lemma 2.3, defining a parametrized analytic continuation of (φ, x_0) along λ. □

Let $\alpha : (\tilde{S}; \tilde{s}_0) \to (S; s_0)$ be a holomorphic map between complex spaces with base points, $\alpha(\tilde{s}_0) = s_0$. Let F be a parametrized analytic continuation of φ along $\Sigma := \mathrm{Graph}(\lambda)$. Let $\mathcal{V} \subset S \times X$ be an open neighborhood of Σ on which F can be defined. Consider $\tilde{\lambda} : (\tilde{S}; \tilde{s}_0) \to (X; x_0)$ for $\tilde{\lambda} := \lambda \circ \alpha$. Then the graph $\tilde{\Sigma} := \mathrm{Graph}(\tilde{\lambda}) \subset \tilde{S} \times X$ is given by $\tilde{\Sigma} = (\alpha, \mathrm{id})^{-1}(\Sigma)$. The meromorphic map $\tilde{F} := (\alpha, \mathrm{id})^* F$ is defined on $\widetilde{\mathcal{V}} := (\alpha, \mathrm{id})^{-1}(\mathcal{V})$, and the germ of meromorphic map $\tilde{F}$ into X' along $\tilde{\Sigma}$ is a parametrized analytic continuation of the germ of meromorphic map φ at x_0 along the map $\tilde{\lambda} : (\tilde{S}, \tilde{s}_0) \to (X, x_0)$. By abuse of notations we will write $\tilde{F} = \alpha^* F$. $(\tilde{F}; \tilde{\Sigma})$ is the *parametrized analytic continuation of* $(\varphi; x_0)$ along $\tilde{\lambda}$ *obtained by pulling back* $(F; \Sigma)$.

The proof of Proposition 2.1 can be easily modified to give:

PROPOSITION 2.5. – *Under the assumptions of Theorem* 1.2, *let* B *be a complex space and* $\beta : B \to \mathcal{K}^o$ *be a holomorphic map, with associated holomorphic* $\mathbb{P}^1$*-bundle* $\hat{\rho} : \mathcal{P} = \beta^* \mathcal{U}^o \to B$ *and induced tautological map* $\hat{\beta} : \mathcal{P} \to \mathcal{U}^o = \rho^{-1}(\mathcal{K}^o)$. *Write* $b_0 \in B$ *resp.* $s_0 \in \hat{\rho}^{-1}(b_0)$ *for chosen distinguished points on* B *resp.* $\mathcal{P}$, *such that* $\mu(\hat{\beta}(s_0)) = x_0$. *Suppose there exists a holomorphic section* $\sigma : B \to \mathcal{P}$ *such that* $\sigma(b_0) = s_0$. *Consider* $\mu \circ \hat{\beta} : (\mathcal{P}; s_0) \to (X; x_0)$ *and* $\mu \circ \hat{\beta}|_{\sigma(B)} : (\sigma(B), s_0) \to (X; x_0)$. *Denote by* $\Sigma \subset \mathcal{P} \times X$ *resp.* $\Sigma_0 \subset \sigma(B) \times X$ *the graphs of* $\mu \circ \hat{\beta}$ *resp.* $\mu \circ \hat{\beta}|_{\sigma(B)}$. *Assume now that there exists a parametrized analytic continuation* $(F_0; \Sigma_0)$ *of* $(\varphi; x_0)$ *along* $\mu \circ \hat{\beta}|_{\sigma(B)}$. *Then, there exists a parametrized analytic continuation* $(F; \Sigma)$ *of* $(\varphi; x_0)$ *along* $\mu \circ \hat{\beta}$ *such that the restriction of* F *to* $\sigma(B) \times X$ *agrees with* F_0 *as germs along* Σ_0.

Proof. – As in Lemma 2.2, F_0 induces $F_0^{\#} : B \to \mathcal{K}'$. Choose a neighborhood $\mathcal{V}$ of Σ_0 where F_0 is defined. We can cover $\mathcal{P}$ by open subsets of the form $\hat{\beta}^{-1}(\mathcal{D}_w)$ where $\mathcal{D}_w \subset \mathcal{U}^o$ is as defined in Lemma 2.4, in such a way that for each $\hat{\beta}^{-1}(\mathcal{D}_w)$, there exists a free rational curve C and a chain of open sets $\hat{\beta}^{-1}(\mathcal{D}_{w_i}), i = 0, 1, \ldots, k$, with $w_i \in \rho^{-1}([C])$ satisfying $w_0 = w, \mathcal{D}_{w_i} \cap \mathcal{D}_{w_{i+1}} \cap \rho^{-1}([C]) \neq \emptyset$ and $\hat{\beta}^{-1}(\mathcal{D}_{w_k}) \cap \sigma(B) \neq \emptyset$. By pulling back the analytic continuation $\tilde{\varphi}$ of φ to $\mathcal{D}_w$ obtained in Lemma 2.4, we can find analytic continuation $\hat{\varphi}$ to $\hat{\beta}^{-1}(\mathcal{D}_w)$. Then $\hat{\varphi}^{\#} = F_0^{\#}$ as germs at the points of B where it is defined. Thus the analytic continuation is uniquely well-defined by Lemma 2.3 and can be patched together to define F. □

3. Adjunction of standard rational curves

Throughout this section, we assume the situation of Theorem 1.2. We say that an irreducible subvariety $A \subset X$ is *saturated* if for any C with $[C] \in \mathcal{K}^o$, either $C \subset A$ or $C \cap A = \emptyset$.

LEMMA 3.1. – *There exists a countable union of proper subvarieties of* X, *so that the only saturated subvariety of* X *containing a point outside this countable union is* X *itself.*

Proof. – Otherwise the union of saturated subvarieties of dimension $< n$ cover a Zariski-open subset of X. Thus there exists an irreducible subvariety $\mathcal{A}$ of the Hilbert scheme of X whose generic point corresponds to a saturated proper subvariety of X so that the members of $\mathcal{A}$ cover the whole X. By choosing a suitable subvariety of $\mathcal{A}$, we get a hypersurface $H \subset X$ which is the closure of the union of some collection of saturated proper subvarieties of X. Choose a $\mathcal{K}^o$-curve C_1 which is not contained in H. From the Picard number condition, C_1 intersects H. Thus small deformations of C_1 intersect generic points of H by Lemma 1.1(1). This gives standard rational curves not contained in H but intersecting saturated subvarieties lying in H, a contradiction to the definition of saturated subvarieties. □

We say that a point $x \in X$ is *generic with respect to saturation* if the only saturated subvariety of X containing x is X itself.

Let $x_0 \in X$ be generic with respect to saturation. Suppose we are given a 4-tuple (S, s_0, V, λ) where S is an irreducible projective variety, V is a nonempty Zariski-open subset of S, s_0 is a point of V, and $\lambda : (S; s_0) \to (X; x_0)$ is a holomorphic map generically finite over its image with $\lambda(s_0) = x_0$. The role to be played by V will be explained in Propositions 3.2 and 3.3 below. If $\lambda(S) \neq X$, we will construct a new 4-tuple $(\hat{S}, \hat{s}_0, \hat{V}, \hat{\lambda})$ satisfying the same conditions as (S, s_0, V, λ) in the following way:

Consider the natural map $\mu:\mathcal{U}\to X$, the pull-back $\lambda^*\mu:\lambda^*\mathcal{U}\to S$ and the tautological map $\beta:\lambda^*\mathcal{U}\to\mathcal{U}$. Since $\lambda(S)$ is not saturated by the choice of x_0, generic fibers of $\lambda^*\mu$ correspond to standard rational curves which do not lie on S. Choose a generic point $u\in\mathcal{U}^o\cap\mu^{-1}(x_0)$ and let $\lambda^*u\in\lambda^*\mathcal{U}$ be the lifting of u lying above s_0. Since $\lambda^*\mathcal{U}$ is projective, there exists an irreducible projective subvariety $E\subset\lambda^*\mathcal{U}$ such that $\lambda^*\mu|_E:E\to S$ is generically finite and $\lambda^*u\in E$. Let $\alpha:Q\to E$ be a normalization of E and $q_0\in Q$ be a point such that $\alpha(q_0)=\lambda^*u$. Then $(\rho\circ\beta\circ\alpha)^*\mathcal{U}$ defines an irreducible variety $\mathcal{P}$ with a natural map $\gamma:\mathcal{P}\to Q$ which is a $\mathbf{P}_1$-bundle over a nonempty Zariski open subset of Q. There is a tautological section $\sigma:Q\to\mathcal{P}$ of γ where $\sigma(q)$ corresponds to the point $\beta\circ\alpha(q)$ of the fiber of $\mathcal{U}$ over $\rho\circ\beta\circ\alpha(q)$. We let $\hat S=\mathcal{P}$, $\hat s_0=\sigma(q_0)$ and $\hat\lambda$ to be the natural map from $\mathcal{P}$ to X induced by μ. Then $\hat\lambda(\hat S)$ is an irreducible subvariety of X containing $\lambda(S)$ as a proper subset because $\lambda(S)$ is not saturated. Since $\dim(\hat S)=\dim(S)+1$, this implies that $\hat\lambda$ is generically finite. Let $Q^*\subset Q$ be the open subset $(\lambda^*\mu\circ\alpha)^{-1}(V)$. We define $\hat V$ to be the Zariski-open subset of $\mathcal{P}|_{Q^*}$ where the fibers of γ corresponds to standard rational curves of X. By our choice of u, the point $\hat s_0$ belongs to $\hat V$.

We say that $(\hat S,\hat s_0,\hat V,\hat\lambda)$ is obtained from (S,s_0,V,λ) by *an adjunction of standard rational curves*. This construction is not unique and depends on the choice of E. From the construction and Proposition 2.5, the following is immediate.

PROPOSITION 3.2. – *Let x_0 be generic with respect to saturation and (S,s_0,V,λ) be a 4-tuple where $s_0\in V\subset S$ is a point of a Zariski-open subset in an irreducible projective variety and $\lambda:(S;s_0)\to(X,x_0)$ is a generically finite morphism over $\lambda(S)\neq X$. Let $(\hat S,\hat s_0,\hat V,\hat\lambda)$ be an adjunction of standard rational curves to (S,s_0,V,λ). If there exists a parametrized analytic continuation of $(\varphi;x_0)$ along $\lambda|_V:(V;s_0)\to(X;x_0)$, then there exists a parametrized analytic continuation of $(\varphi;x_0)$ along $\hat\lambda|_{\hat V}:(\hat V;\hat s_0)\to(X;x_0)$.*

Starting from x_0, we can repeatedly apply this construction to obtain:

PROPOSITION 3.3. – *Let $x_0\in X$ be generic with respect to saturation. Then for $1\leqslant k\leqslant n=\dim X$, there exist a k-dimensional irreducible projective variety $S^{(k)}$, a non-trivial Zariski-open subset $V^{(k)}\subset S^{(k)}$, a point $s_0^{(k)}\in V^{(k)}$, and a holomorphic map $\lambda^{(k)}:(S^{(k)};s_0^{(k)})\to(X;x_0)$ generically finite over its image, such that, for any germ $(\varphi;x_0)$ of meromorphic map into X' preserving varieties of rational tangents, there exists a parametrized analytic continuation of $(\varphi;x_0)$ along $\lambda^{(k)}|_{V^{(k)}}:(V^{(k)};s_0^{(k)})\to(X;x_0)$.*

Proof. – To start with, choose a $\mathcal{K}^o$-curve C passing through x_0. Let $S^{(1)}=V^{(1)}=\mathbf{P}_1$ and $\lambda^{(1)}$ be the normalization $\mathbf{P}_1\to C$ with $s_1\in S^{(1)}$ a point over x_0. This satisfies the required analytic continuation property by Proposition 2.1. Now apply Proposition 3.2 inductively to construct $(S^{(k+1)},s_0^{(k+1)},V^{(k+1)},\lambda^{(k+1)})$ as $(\hat S^{(k)},\hat s_0^{(k)},\hat V^{(k)},\hat\lambda^{(k)})$ by an adjunction of standard rational curves. $\square$

Using the above construction, we want to extend the given map φ to a multi-valued meromorphic map defined on a Zariski dense open subset of X, in other words, a meromorphic map defined on an unramified cover of a Zariski open subset of X. Given an unramified morphism $\chi:Z\to X$ from a complex manifold Z and $z\in Z$, we identify $T_z(Z)$ with $\chi^*T_{\chi(z)}(X)$ canonically and define $\mathcal{C}_z$ to be $[d\chi_z]^{-1}\mathcal{C}_{\chi(z)}\subset\mathbf{P}T_z(Z)$.

PROPOSITION 3.4. – *Let $x_0\in X$ be a generic point, $\dim(X)=n$. Then, there exists an n-dimensional normal projective variety $\bar\Sigma$ with a distinguished point $\sigma_0\in\bar\Sigma$, a generically finite holomorphic map $\chi:\bar\Sigma\to X$, $\chi(\sigma_0)=x_0$, and a non-empty smooth Zariski-open subset $Z\subset\bar\Sigma$ such that, writing $\pi=\chi|_Z$:*

(a) *$\pi:Z\to X-D$ is unramified for some divisor D;*

(b) *for any open neighborhood U of x_0 in X, and any meromorphic map $\varphi : U \to X'$ preserving varieties of rational tangents, there exists a meromorphic map $\psi : Z \to X'$ preserving varieties of rational tangents in the sense $\psi_*(\mathcal{C}_z) = \mathcal{C}'_{\psi(z)}$ at points $z \in Z$ at which ψ is locally biholomorphic, such that for some open neighborhood W of σ_0 on $\bar{\Sigma}$ for which $\chi(W) \subset U$, we have $\psi \equiv \chi^*\varphi$ on $W \cap Z$.*

Note that x_0 may lie on D.

Proof. – For $k = n$ in Proposition 3.3, $\lambda^{(n)} : (S^{(n)}; s_0^{(n)}) \to (X; x_0)$ is a surjective generically finite morphism. Write $\lambda = \lambda^{(n)}$, etc. and assume without loss of generality that S is normal. We have a non-empty Zariski-open subset $V \subset S$, $s_0 \in V$, such that, for any germ $(\varphi; x_0)$ of meromorphic map into X' preserving varieties of rational tangents, there exists a parametrized analytic continuation of $(\varphi; x_0)$ along $\lambda|_V : (V; s_0) \to (X; x_0)$. We need to extract a meromorphic map out of this parametrized analytic continuation.

Write $\Sigma \subset V \times X$ for $\mathrm{Graph}(\lambda\big|_V)$ and $(F; \Sigma)$ for the parametrized analytic continuation of $(\varphi; x_0)$ along $\lambda|_V$. Let $pr_X : V \times X \to X$ be the natural projection and $\chi = pr_X|_\Sigma$. Let $\bar{\Sigma}$ be a suitable projective variety compactifying Σ so that χ can be extended to a holomorphic map $\chi : \bar{\Sigma} \to X$. Let $Z \subset \Sigma$ be a smooth Zariski-open set so that χ is unramified on Z. Write $\psi = F|_Z$. At any point $(s, \lambda(s)) \in Z \subset \Sigma$ the germ of $F|_{\{s\}\times X}$ at $(s, \lambda(s))$ preserves varieties of rational tangents at generic points, when $\{s\} \times X$ is identified with X canonically. By the condition (a) of the definition of parametrized analytic continuation, the germ of F at $(s, \lambda(s))$ is of the form $pr_X^*\nu$ for some germ ν at $\lambda(s)$, where ν preserves varieties of rational tangents. It follows that $\psi : Z \to X'$ is a meromorphic map preserving varieties of rational tangents. □

4. Global extension of a meromorphic map

In this section, we will finish the proof of Theorem 1.2. Starting with the unramified covering $\pi : Z \to X - D$ of Proposition 3.4, first we are going to construct a meromorphic map Φ from X to X' extending a given germ of meromorphic map $(\varphi; x_0)$ preserving varieties rational tangents, and then show that Φ is biholomorphic. There are two problems for the construction of Φ: Z is not univalent and the meromorphic map $\psi : Z \to X'$ may have essential singularities along D.

PROPOSITION 4.1. – *In the notation of Proposition* 3.4, *let $(\varphi; x_0)$ be any germ of meromorphic map into X' preserving varieties of rational tangents, and $\psi : Z \to X'$ be the meromorphic map arising from $(\varphi; x_0)$ by parametrized analytic continuation. Let $x \in X - D$ and $z_1, z_2 \in V$ be two points lying above x, i.e., $\pi(z_1) = \pi(z_2) = x$. Then, the germs of meromorphic maps $(\psi; z_1)$ and $(\psi; z_2)$ agree in the sense that $(\psi; z_1) = (\pi^*\xi; z_1)$, $(\psi; z_2) = (\pi^*\xi; z_2)$ for some germ of meromorphic map $(\xi; x)$ at x into X'.*

Proof. – Introduce an equivalence relation on Z by writing $z_1 \sim z_2$ whenever (i) $\pi(z_1) = \pi(z_2)$ and (ii) for each germ $(\varphi; x_0)$ the germs of the extended map $(\psi; z_1)$, resp. $(\psi; z_2)$ at z_1 resp. z_2 agree with each other. Write $\tilde{Z} = Z/\sim$. Then the canonical map $Z \to \tilde{Z}$ and the associated covering $\tilde{\pi} : \tilde{Z} \to X - D$ are unramified. Replacing Z by $\tilde{Z}$ and $\pi : Z \to X - D$ by $\tilde{\pi} : \tilde{Z} \to X - D$ we may assume without loss of generality that given $z_1 \neq z_2$ with $\pi(z_1) = \pi(z_2)$, there exists some germ $(\varphi; x_0)$ of meromorphic map into X' preserving varieties of rational tangents so that the extended map ψ has distinct germs at z_1 and z_2. For this new meaning of Z, Proposition 4.1 amounts to saying that π is bijective. Note that this new Z can still be compactified to a complete algebraic variety to which π can be extended as a generically finite morphism over X.

We need the following lemma which holds for any Fano manifold with Picard number 1.

LEMMA 4.2. – *Let $\pi : Y \to X$ be a generically finite morphism from a normal irreducible variety Y onto a Fano manifold X with Picard number* 1. *Suppose for a generic standard rational curve $C \subset X$ belonging to a chosen standard component $\mathcal{H}^o$, each component of the inverse image $\pi^{-1}(C)$ is birational to C by π. Then $\pi : Y \to X$ itself is birational.*

Proof. – Suppose π is not birational. Since X is simply connected (e.g. [9]), there exists a ramification divisor $R \subset Y$ of π so that $\pi(R)$ is a divisor in X. By genericity of C, we may assume that $\pi^{-1}(C)$ lies on the smooth part of the normal variety Y. Let C_1 be any irreducible component of $\pi^{-1}(C)$ which is birational to C by π. Let $h : \mathbf{P}_1 \to C_1$ be the normalization. Then $\pi \circ h$ is the normalization of C. Thus a deformation $h_t : \mathbf{P}_1 \to Y$ of C_1 induces a deformation $\pi \circ h_t$ of C. On the other hand, by the genericity of C, taking pre-images with respect to π any small deformation of C can be lifted to a small deformation of C_1. It follows that the space of deformations of C and the space of deformations of C_1 have equal dimensions. So we have $K_Y \cdot C_1 = K_X \cdot C$ (cf. [10], II.1.2). This implies C_1 is disjoint from the ramification divisor $R \subset Y$. Since this holds for any component C_1 of $\pi^{-1}(C)$, C is disjoint from the divisor $\pi(R)$, a contradiction to the assumption that X is of Picard number 1. □

Now we prove that π is bijective. Suppose not. By Lemma 4.2, for a standard rational curve C intersecting U, there exists an irreducible quasi-projective curve C^* on Z such that $\pi(C^*) \subset C$ and $\pi|_{C^*}$ is not birational. For a generic point $x \in C \cap U$, we have $z_1 \neq z_2$ on C^* such that $\pi(z_1) = \pi(z_2) = x$. We can find a germ φ of meromorphic map to X preserving rational tangents so that the germs $(\psi; z_1)$ and $(\psi; z_2)$ obtained by analytically continuing φ are distinct. Choose an arc Υ on C^* starting from z_1 ending at z_2. The analytic continuation of $(\psi; z_1)$ along Υ gives $(\psi; z_2)$. However there is an analytic continuation of φ along C by Proposition 2.1. So the analytic continuation along the loop $\pi(\Upsilon)$ on C must give the same germ at x. The analytic continuation along C^* should agree with the one pulled back from C via $\pi|_{C^*}$. Thus follows $(\psi; z_1) = (\psi; z_2)$, a contradiction. □

PROPOSITION 4.3. – *Any germ of meromorphic map $(\varphi; x_0)$ to X' preserving varieties of rational tangents extends to a meromorphic map from X to X'.*

Proof. – From Proposition 4.1, we see that there exists a Zariski-open set $X^o \subset X$ such that any germ of meromorphic map $(\varphi; x_0)$ to X' preserving varieties of rational tangents extends to a meromorphic map Φ from X^o to X'. Suppose there exists a divisor $D \subset X - X^o$. Since X is of Picard number 1, we have a $\mathcal{K}^o$-curve C through a generic point b of D by Lemma 1.1(1). Pick an irreducible branch of the germ of C at b. Then by Proposition 2.1, we can extend Φ to the union of X^o and a neighborhood U_b of b. Applying this to each codimension 1 component of $X - X^o$, Φ can be extended outside a codimension > 1 set, and we are done by Hartogs extension for meromorphic maps. □

Let $\Phi : X \to X'$ be the meromorphic map in Proposition 4.3. Since Φ preserves varieties of rational tangents, the strict transform of $\mathcal{C} \subset \mathbf{P}T(X)$ by Φ_* must be a component of $\mathcal{C}'$, and must agree with $\mathcal{C}'$ from the irreducibility of $\mathcal{C}'$. It follows that $\varphi_*(\mathcal{C}_x) = \mathcal{C}'_{\varphi(x)}$ for generic $x \in U$. This means that $\varphi^{-1} : U' \to U$ preserves varieties of rational tangents. Now applying Proposition 4.3 to φ^{-1} which is a germ of meromorphic map at $\varphi(x_0) \in X'$ to X preserving varieties of rational tangents, we get a meromorphic map $\Phi^{-1} : X' \to X$. Thus Theorem 1.2 follows from the ensuing proposition whose proof is given in 3.2.5 of [5]. Although the proof there is stated in the case when both X and X' are irreducible Hermitian symmetric spaces, it works verbatim in the general case. Here we will give a simplified proof.

PROPOSITION 4.4. – *Let X, X' be as in Theorem* 1.2 *and $\Phi : X \to X'$ be a birational map preserving varieties of rational tangents. Then Φ is biholomorphic.*

Proof. – We denote by $B \subset X$ the subvariety on which Φ fails to be a local biholomorphism and call B the bad locus of Φ. We claim that Proposition 4.4 will follow if we show that B is of codimension $\geqslant 2$. Since X and X' are Fano we may choose k large enough so that both K_X^{-k} and $K_{X'}^{-k}$ are very ample. Let s be a pluri-anticanonical section on X' in $\Gamma(X', K_{X'}^{-k})$. Then $\Phi^* s$ is a well-defined pluri-anticanonical section on $X - B$. It extends across B under the assumption that B is of codimension $\geqslant 2$. It follows that Φ induces a linear monomorphism $\theta : \Gamma(X', K_{X'}^{-k}) \to \Gamma(X, K_X^{-k})$ and hence a linear isomorphism $\theta^* : \Gamma(X, K_X^{-k})^* \to \Gamma(X', K_{X'}^{-k})^*$ by taking adjoints. Identifying X resp. X' as a complex submanifold of $\mathbf{P}\Gamma(X, K_X^{-k})^*$ resp. $\mathbf{P}\Gamma(X', K_{X'}^{-k})^*$, Φ is nothing other than the restriction of the projectivization $[\theta^*] : \mathbf{P}(\Gamma(X, K_X^{-k})^*) \to \mathbf{P}(\Gamma(X', K_{X'}^{-k})^*)$ to X, thus a biholomorphism.

It remains to show that the bad locus B of $\Phi : X \to Y$ is of codimension $\geqslant 2$. Otherwise let $R \subset B$ be an irreducible component of codimension 1 in X. The strict image $\Phi(R)$ has codimension $\geqslant 2$ in X'. Since X has Picard number 1, all $\mathcal{K}^o$-curves intersect R. Thus their images under Φ will intersect $\Phi(R)$. But these images are generic $\mathcal{H}'$-curves by Proposition 1.3, a contradiction to Lemma 1.1(3). □

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